Engineering Problem Involving Diophantine Algebra

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Diophantine algebra (approximately 250 AD), where solutions to any given problem are restricted (fascinatingly, perhaps, but rather arbitrarily, or so it would seem) to nonnegative integer numbers, is a quaint if fascinating branch of mathematics, which makes its occasional appearance nowadays in puzzles and brainteasers, but practically never in engineering science. It was therefore with some interest that, while studying a problem having to do with so mundane a subject as the devolatilization of polymers, we convinced ourselves that the problem is a nice example of diophantine algebra arising in the common sense world of today.

The problem arises in slit devolatilization. The process, if considered in the restricted sense of a single slit, can be modeled (Maffettone et al., 1991) rather successfully, and even when the polymeric phase is regarded as a Newtonian fluid, it has some rather interesting features: the pressure drop vs. flow rate plot has the shape shown in Figure 1. The low flow rate asymptote is a straight line through the origin having the slope calculated by taking the viscosity of the solution in equilibrium with the exit conditions (which are very high because of the very low concentration of the volatile component), while the high flow rate asymptote (again a straight line going through the origin) corresponds to the low viscosity of the feed solution.

The pressure drop is a unique function of the flow rate Q, $f(Q)$, but the converse is obviously not true. This implies that in an industrial multislit unit where the number of slits is typically quite large (of the order of 1,000 or more), one may run into the problem of some slits working in a high flow rate mode where devolatilization is extremely inefficient. The technical problem has been discussed in detail by Ianniruberto et al. (1993, 1996). In this article, we analyze the conceptually interesting problem where, with the $f(Q)$ shape in Figure 1, there are more than one conduits in parallel (which we call slits) subject to two conditions: the pressure drop must be the same in all slits, and the total flow rate is assigned.

Region of Instability

Let there be N slits in parallel, and let Q_T be the total (volumetric) flow rate, so that $Q^* = Q_T/N$ is the average flow rate per slit. We call globally stable a situation where $Q = Q^*$ in all slits, and no other distribution of flow rates is locally stable (in the sense of being stable to infinitesimal perturbations). Clearly, if $Q^* < P$, and if $Q^* > T$, only a globally stable solution of the mass balance and pressure drop equations exists. Hence, we define as the region of instability the one where $P < Q^* < T$. Three subcases need to be analyzed

case I, $P < Q^* < R$;
case II, $S < Q^* < T$;
case III, $R < Q^* < S$.

Case I

$Q^* - f(Q^*)$ is a locally stable point (Ianniruberto et al., 1993), and thus two different possibilities of locally stable solutions can be envisaged. First, it is possible that all slits work at $Q^* - f(Q^*)$. There is no possibility that M slits may work at Q^*, and the remaining slits split themselves into two subsets working at different flow rates. This is because the latter should work at a pressure drop $f(Q^*)$, and there is only one flow rate larger than Q^* satisfying this condition with none at a lower flow rate to compensate for it. Hence, the case of interest is the one where no slit works at Q^*, and the N slits break up in two subsets m and n, with

$$m + n = N \tag{1}$$

with m slits discharging a flow rate Q', and n slits discharging a flow rate Q'', and

$$mQ' + nQ'' = NQ^* \tag{2}$$

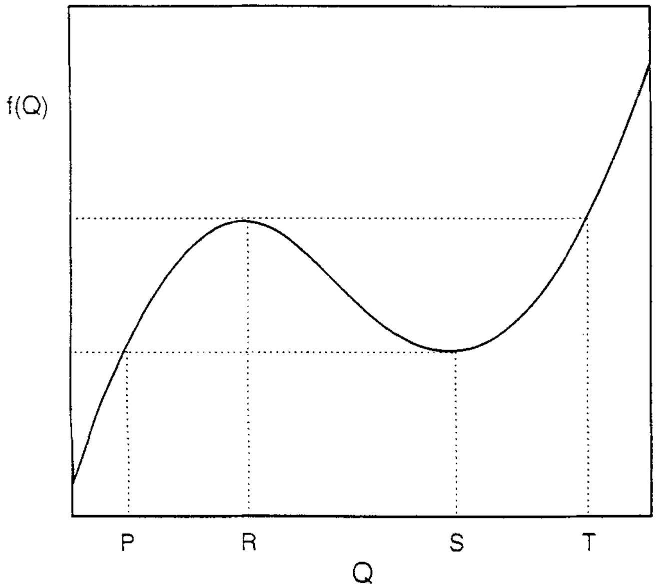

Figure 1. The pressure drop vs. flow rate curve.

The only significant parameters are the values of P, R, S and T.

The pressure drop through all the slits must be the same; this is why the N slits split into at most two subsets

$$f(Q') = f(Q'') \tag{3}$$

Notice that Q' and Q'' have to be locally stable points, and, hence, Eq. 3 sets the conditions $P < Q' < R$, $S < Q'' < T$. Equation 2 implies that $Q' < Q^*$, and, therefore,

$$f(Q') = f(Q'') < f(Q^*).$$

An example of such a situation is indicated by the dotted line in Figure 2. Equation 2 implies that a lever rule applies, say

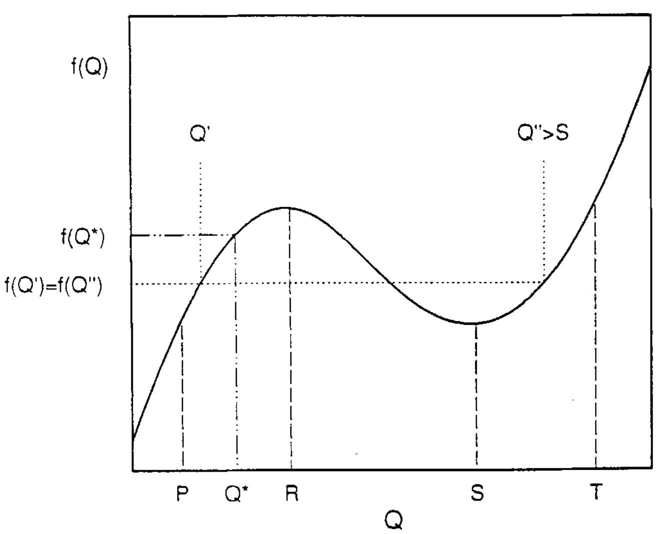

Figure 2. Case I.

$$(Q'' - Q^*)/(Q^* - Q') = m/n \tag{4}$$

Now notice that m/n is a rational number, restricted by condition 1, which is what lends the diophantine flavor to the problem. We will call a rational number m/n with both m and n positive integers constrained by Eq. 1 an N-order rational number (N-ORN). It is easy to establish the upper and lower bounds of the N-ORN m/n

$$1/(N-1) \leq m/n \leq N-1 \tag{5}$$

where the two bounds correspond respectively to $m = 1$ and $n = 1$.

The quantity $(Q'' - Q^*)/(Q^* - Q')$ has no upper bound (it approaches ∞ if Q' approaches Q^*), but it has a lower bound

$$(Q'' - Q^*)/(Q^* - Q') > (S - Q^*)/(Q^* - P) \tag{6}$$

It follows at least one solution other than $Q = Q^*$ in all slits that exist provided that

$$N > (S - P)/(Q^* - P) \tag{7}$$

or equivalently

$$Q^* > [(N-1)P + S]/N > P \tag{8}$$

The condition in Eq. 8 is given in terms of quantities which are uniquely determined by the $f(Q)$ curve. Notice that $[(N-1)P + S]/N$ may well exceed R, in which case the whole low flow rate branch of the curve would correspond to globally stable values of Q^*. This situation occurs if $N < (S - P)/(R - P)$, which of course is possible only if $P + S > 2R$.

When N approaches ∞, the condition in Eq. 8 is guaranteed to be satisfied, since $Q > P$ for case I. For any finite value of N, the condition in Eq. 8 establishes an upper bound for a globally stable Q^* which is larger than P. *In other words, the diophantine nature of the problem entails that a globally stable situation with all slits working at $Q = Q^*$ may exist at Q^* values inside the region of instability.* Notice that, as N approaches ∞ (so that an N-ORN may be as small as one wishes), the condition in Eq. 8 is guaranteed to be satisfied for any Q^* in the region of instability.

We now come to the question of whether more than one double solution may exist. It is obvious that for K ($< N$) solutions to exist the condition to be satisfied is

$$Q^* > [(N-K)P + KS]/N \tag{9}$$

However, since $Q^* < R$ for case I, the condition in Eq. 9 may well be impossible to satisfy for large values of K. Define α_K as

$$\alpha_K = K/N \tag{10}$$

With this, the condition in Eq. 9 becomes

$$Q^* > (1 - \alpha_K)P + \alpha_K S \tag{11}$$

Then K distinct nontrivial solutions exist if

$$R > (1-\alpha_K)P + \alpha_K S \tag{12}$$

while $K+1$ nontrivial solutions do not exist if

$$R < (1-\alpha_{K+1})P + \alpha_{K+1} S \tag{13}$$

The conditions in Eqs. 12–13 determine the diophantine problem of the maximum number K_{MAX} of nontrivial locally stable solutions existing in case I (incidentally, this and analogous results in the following are the only occasion where we actually solve a diophantine problem; the solution is obtained rather trivially without making use of classical congruence methods).

As N approaches ∞, so does K_{MAX}: in this limit, the diophantine nature of the problem manifests itself only in that the number of different solutions becomes countably (rather than continuously) infinite.

Case II

This is the mirror image of Case I, as shown in Figure 3 (again two possibilities arise since Q^* is a locally stable point with only the two-subsets case in need of analysis). Equations 4 and 5 still apply, but this time $(Q''-Q^*)/(Q^*-Q')$ has no lower bound (it approaches zero when Q'' approaches Q^*). However, it has an upper bound:

$$(Q''-Q^*)/(Q^*-Q') < (T-Q^*)/(Q^*-R) \tag{14}$$

Except for the case where all slits work at Q^*, the pressure drop will be larger than $f(Q^*)$. At least one double solution exists provided that

$$N > (T-R)/(T-Q^*) \tag{15}$$

or, equivalently,

$$Q^* < [(N-1)T+R]/N < T \tag{16}$$

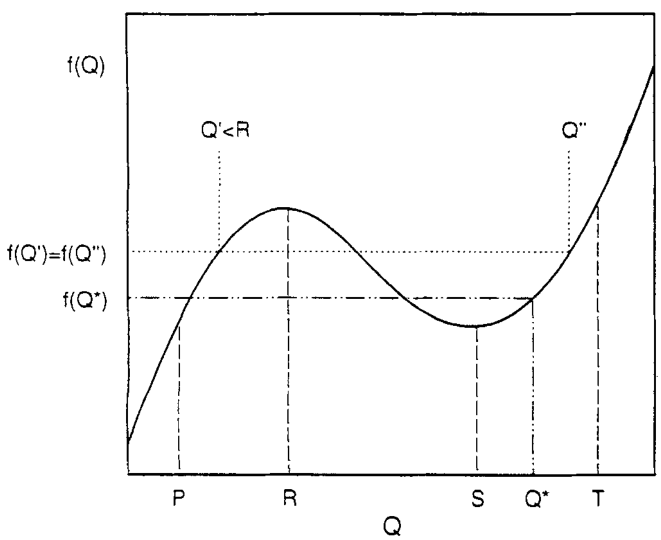

Figure 3. Case II.

The whole high flow rate branch of the curve corresponds to globally stable values of Q^* if $N < (T-R)/(T-S)$, which, of course, can only happen if $T+R < 2S$.

K distinct double solutions exist provided that

$$Q^* < (1-\alpha_K)T + \alpha_K R \tag{17}$$

However, in Case II, $Q^* > R$, and, hence, K distinct double solutions exist, but $K+1$ does not, if the following two conditions are satisfied

$$S < (1-\alpha_K)T + \alpha_K R \tag{18}$$

$$S > (1-\alpha_{K+1})T + \alpha_{K+1} R \tag{19}$$

Case III

Since in this case $Q^* - f(Q^*)$ is a locally unstable point (Iannirubert o et al., 1993), the only possibility of a locally stable solution is that of a two-way split, as shown in Figure 4. However, as will be seen, there may be situations where even a two-way split is impossible, such as situations where no locally stable solution exists.

For simplicity, consider first the case where $N=2$, so that the only two-way split possible is $m=n=1$, and Eq. 4 gives

$$(Q''-Q^*)/(Q^*-Q') = 1 \tag{20}$$

The quantity $(Q''-Q^*)/(Q^*-Q')$ has now both an upper and a lower bound

$$(S-Q^*)/(Q^*-P) < (Q''-Q^*)/(Q^*-Q') < (T-Q^*)/(Q^*-R) \tag{21}$$

As Q^* approaches S, the quantity $(T-Q^*)/(Q^*-R)$ approaches $(T-S)/(S-R)$, and this may be less than unity, so that the second part of the condition in Eq. 21 may well be impossible to satisfy. Conversely, as Q^* approaches R, the quantity $(S-Q^*)/(Q^*-P)$ approaches $(S-R)/(R-P)$

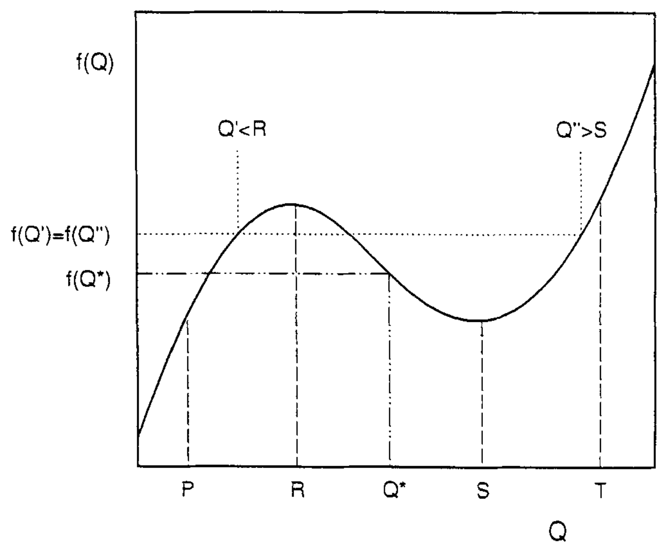

Figure 4. Case III.

which may well be larger than unity, so that the first part of the condition in Eq. 21 may also be impossible to satisfy. This shows that indeed there are situations where no locally stable solution exists. The results so far can be stated as follows:

Theorem A. In case III with $N = 2$, if $(T - S)/(S - R) > 1$ and $(S - R)/(R - P) < 1$, at least one locally stable double solution exists for all values of Q^*.

When N is larger than 2, so that the N-ORN m/n has the upper and lower bounds given in Eq. 5, the conditions for existence of at least one locally stable double solution become weaker:

Theorem B. In Case III, if $(T - S)/(S - R) > 1/(N - 1)$ and $(S - R)/(R - P) < N - 1$, at least one locally stable double solution exists for all values of Q^*.

Here, of course, theorem A is a special case of theorem B. Notice that, when N approaches ∞, theorem B becomes empty, since its two conditions are necessarily satisfied.

It is interesting to observe that the following partial converse of theorem B holds:

Theorem C. If either one of the conditions of theorem B is not satisfied, there exist values of Q^* which admit no locally stable solution.

Theorem C is a consequence of the diophantine nature of the problem: should the quantity m/n be allowed to take any nonnegative value, without the restriction of being an N-ORN, at least one locally stable solution would always be available.

If the first condition of theorem B is not satisfied, so that $S > [T(N-1) + R]/N$, at least one locally stable double solution exists provided that

$$Q^* < [R + (N-1)T]N < S \tag{22}$$

and, if the second condition is not satisfied, at least one locally stable double solution exists if

$$Q^* > [S + (N-1)P]/N > R \tag{23}$$

Again, note that as N approaches ∞, the conditions in Eqs. 22 and 23 approach the conditions which define Case III.

Suppose that the first condition of theorem B is not satisfied, so that at least one locally stable double solution exists only if the condition in Eq. 22 is satisfied. For K solutions to exist, the following condition must be satisfied

$$Q^* < (1 - \alpha_K)T + \alpha_K R \tag{24}$$

which is more stringent than the condition in Eq. 22 unless α_K has its minimum value of $1/N$. $K + 1$ locally stable double solutions cannot exist if

$$Q^* > (1 - \alpha_{K+1})T + \alpha_{K+1} R \tag{25}$$

The mirror image of this is when the second condition of theorem B is not satisfied, where K solutions exist but $K + 1$ solutions do not exist if

$$Q^* > (1 - \alpha_K)P + \alpha_K S \tag{26}$$

$$Q^* < (1 - \alpha_{K+1})P + \alpha_{K+1} S \tag{27}$$

Engineering Significance

In the specific case of multislit devolatilization of polymers, one wants to work at as large a value of Q^* as possible on the low flow rate branch of the curve where devolatilization is efficient, with no slit working at the high flow rate branch. This would seem to imply that the largest possible value of Q^* is P, but in fact the analysis given above shows that the largest possible value of Q^* is $[(N-1)P + S]N$, as shown in Eq. 8. The advantage, however, is only marginal, since a large number of slits is used in industrial units. With $N = 1{,}000$ (a typical industrial unit value), $S/P = 25$ (corresponding to the lowest curve in Figure 2 of Iannirubert et al., 1993), and the largest stable Q^* exceeds P by only 2.4%.

However, since for the same curve $(S - P)/(R - P) \approx 15$, up to 14 slits in parallel could work at globally stable conditions with Q^* up to R. If 1,000 slits are needed working at $Q^* = P$, only 400 would be needed working at $Q^* = R$, since for the same case $R/P \approx 2.5$. Hence, an industrial unit with 30 independent feed chambers, each one feeding 14 slits, could efficiently work at Q^* close to R on a total of 420 slits in parallel. This is significantly better than either having 400 independent feed chambers, or a single feed chamber with 1,000 slits working at a Q^* value only marginally in excess of P.

More generally, two interesting properties emerge from the analysis given above. First, there are conditions where, at sufficiently small values of N, all the locally stable values of Q^* on one or both branches of the curve become globally stable (see the discussion following Eqs. 8 and 16): this implies that the actual operation of a parallel conduits unit (such that the $f(Q)$ curve has the shape in Figure 1) may be significantly more stable than a crude analysis would suggest. Secondly (and somewhat conversely), the possible existence of conditions such that no locally stable solution exists suggests that periodic behavior could be observed as the only mode of operation of a parallel conduits industrial unit.

Finally, it is perhaps useful to discuss a generalization of the problem considered here. Whenever one feeds in parallel N identical units, each one of which admits at least two different locally stable steady-state solutions (the units might be CSTRs), it is possible that the N units split themselves into m units working at one steady state and n working at the other one, with m, n and N satsfying Eq. 1. Any such situation again reduces to a diophantine problem.

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